UNCERTAINTY OF THE MEASUREMENT

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1. INTRODUCTION

Needs for evaluation uncertainty of the measurement is old as a human need for measurement.How we now absolutely exact measurement don't exist, but there are something which surely exist that is uncertainty of the measurement in calibration.The common interest all of as who treat measurement problems is how to establishes general rules for evaluating and expressing uncertainty in measurement that can be followed in most fields of physical measurements.The document which I treat concentrates on the method most suitable for the measurements in calibration laboratories and describes an unambiguous and harmonized way of evaluating and stating the uncertainty of measurement.

It comprises the following subjects:

- definitions basic to the document
- relationship between the uncertainty of measurement of the input quantity and the uncertainty of measurement of the output quantities
- methods for evaluating the uncertainty of measurements of input quantities
- expanded uncertainty of measurement of the output quantity
- statement of the uncertainty of measurement
- a step by step procedure for calculating the uncertainty of measurement

EAL defines the best measurement capability as the smallest uncertainty of measurement that a laboratory can achieve within its scope of accreditation,when performing more or less routine calibrations of nearly ideal measuring instruments designed for the measurement of that quantity.

1.1. Outline and definitions

1.1. The statement of the result of measurement is complete only if it contains both the value attributed to the measured and the uncertainty of measurement associated with that value.All quantities which are not exactly known are treated as random variables, including the influence quantities which may affect the measured value.

1.2 The uncertainty of measurement is a parameter ,associated with the result of a measurement, that characterizes the dispersion of the values that could reasonably be attributed to the measurand.The shorthand term uncertainty is used for uncertainty of measurement if there is no risk of misunderstanding.

1.3 The measurands are the particular quantities subject to measurement.In calibration one usually deals with only one measured or output quantity Y that depends upon a number of input quantities X_i (i=1,2,….N) according to the functional relationship.

$$Y=f(X_1,X_2,\ldots X_N) \qquad (1.1)$$

The model function f represents the procedure of the measurement and the method of evaluation.It describes how values of the output quantity Y are obtained from values of the input quantities X_i .In most cases it will be an analytical expression ,but it may also be a group of such expression which include corrections and correction factors for systematic effects ,thereby leading to a more complicated relationship that is not written down as one function explicitly.Further,f may be determined experimentally,or exit only as a computer algorithm that must be evaluated numerically, or it may be a combination of all of these.

1.4 The set of input quantities X_i may be grouped into two categories according to the way in which the value of the quantity and its associated uncertainty have been determined (a)quantities whose estimate and associated uncertainty are directly determined in the current measurement.These values may be obtained, for example,from a single observations,or judgement based on experience.They may involve the determination of corrections to instrument readings as well as corrections for influence quantities, such as ambient temperature ,barometric pressure or humidity:
(b)quantities whose estimate and associated uncertainty are brought into the measurement from external sources ,such as quantities associated with calibrated measurement standards, certified reference materials or reference data obtained from handbooks.

1.5. An estimate of the measured Y ,the output estimate denoted by y,is obtained from equation (1.1) using input estimates x_I for the values of the input quantities X_i

$$y=f(x_1,x_2,\ldots,x_N) \qquad (1.2)$$

$(x_1,x_2,\ldots,x_N)$ are best estimates that have been corrected for all effects significant for the model.If not,the necessary corrections have been introduced as separate input quantities.

1.6 Standard deviation is used as a measure of the dispersion of values.The standard uncertainty of measurement associated with the output estimate or measurement result y ,denoted by u(y) ,is the standard deviation of the measured Y.It is to be determined from the estimates x_I of the input quantities X_i and their associated standard uncertainties $u(x_i)$.The standard uncertainty associated with an estimate has the same dimension as the estimate.The relative standard uncertainty of measurement may be appropriate which is the standard uncertainty of measurement associated with an estimate divided by the modules of that estimate and is therefore dimensionless .This concept can not be used if the estimate equals zero.

2. CALCULATION OF THE STANDARD UNCERTAINTY OF THE OUTPUT ESTIMATE

2.1 For uncorrelated input quantities the square of the standard uncertainty associated with the output estimate y is given by:

$$u^2(y)=\sum_{i=1}^{N} u_i^{\,2}(y) \qquad (2.1)$$

The quantity u_i (y) (i=1,2,….,N) is the contribution to the standard uncertainty associated with the input estimate y resulting from the standard uncertainty associated with the output estimate x_i.

$$u_i(y)=c_i u(x_i) \qquad (2.2)$$

c_i is the sensitivity coefficient associated with the input estimate x_I of the output x_j.

2.2 The sensitivity coefficient c_i describes the extent to which the output estimate y is influenced by variations of the input estimate x_i .It can be evaluated from the model function f by equation

$$c_i=\frac{\partial f}{\partial x_i}=\frac{\partial f}{\partial X_i} \qquad X_i=x_i \ldots X_N=x_N \qquad (2.3)$$

or by using numerical methods,i.e by calculating the change in the output estimate y due to a change in the output estimate y due to a change in the input estimate x_i of $+u(x_i)$ and - $u(x_i)$ taking as the value of c_i the resulting difference in y divided by $2u(x_i)$.Sometimes it may be more appropriate to find the change in the output estimate y from an experiment by repeating the measurement at e.g. $\pm\ u(x_i)$.

2.3 Whereas $u(x_i)$ is always positive, the contribution $u_i(y)$ according to equation (2.2) is either positive or negative ,depending on the sign of the sensitivity coefficient c_i.

2.4 If the model function f is a sum or difference of the input quantities X_i

$$f(X_1,X_2.........X_N)=\sum_{i=1}^{N} p_i \qquad (2.4)$$

the output estimate according to equation (1.2) is given by the corresponding sum or difference of the input estimates

$$y=\sum_{i=1}^{N} p_i X_i \qquad (2.5)$$

whereas the sensitivity coefficients equal p_i and equation (2.1) converts to

$$u^2(y)=\sum_{i=1}^{N} p_i^{\,2}\ u^{\,2}(x_i) \qquad (2.6)$$

2.5 If the model function f is a product or quotient of the input quantities X_i

$$f(X_1,X_2......X_N)=c\prod_{i=1}^{N} X_i^{\,pi} \qquad (2.7)$$

the output estimate again is the corresponding product or quotient of the input estimates

$$y = c\prod_{i=1}^{N} X_i^{p_i} \qquad (2.8)$$

The sensitivity coefficients equal $p_i y/x_i$ in this case and an expression analogous to equation (2.6) is obtained from equation (2.1) ,if relative standard uncertainties $w(y)=u(y)/|y|$ i $w(x_i)=u(x_i)/|x_i|$ are used:

$$w^2(y)=\sum_{i=1}^{N} p_i^{\,2} w^{\,2}(x_i) \qquad (2.9)$$

2.6 If two input quantities X_i and X_k are correlated to some degree,i.e. if they are mutually dependent in one way or another,their covariance also has to be considered as a contribution to the uncertainty.The ability to take into account the effect of correlation's depends on the knowledge of the measurement process and on the judgement of mutual dependencies of the input quantities.In general it should be kept in mind that neglecting correlation's between input quantities can lead to an incorrect evaluation of the standard uncertainty of the measured.

2.7 The covariance associated with the estimates of two input quantities X_i and X_k may be taken to be zero or treated as insignificant if

(a) the input quantities are independent, for example, because they have been repeatedly but not simultaneously observed in different independent experiments or because they represent resultant quantities of different evaluations that have been made independently, or if
(b) either of the input quantities X_i and X_k can be treated as constant, or if
(c) investigation gives no information indicating the presence of correlation between the input quantities X_i and X_k.

Sometimes correlation's can be eliminated by a proper choice of the model function:

2.8 The uncertainty analysis for a measurement should include a list of all sources of uncertainty together with the associated standard uncertainties of measurement and the methods of evaluating them.For repeated measurements the number n of observations also has to be stated.For the sake of clarity, it is recommended to present the data relevant to this analysis in the , form of a table.In this table all quantities should be referenced by a physical symbol X_i or a short identifier.For each of them at least the estimate x_i, the associated standard uncertainty of measurement $u(x_i)$, the sensitivity coefficient c_i and the different uncertainty contributions $u_i(y)$ should be specified.The dimension of each of the quantities should also be stated with the numerical values given in the table.

2.9 A formal example of such an arrangement is given as table 2.1 applicable for the case of uncorrelated input quantities.The standard uncertainty associated with the measurement result u(y) given in the bottom right corner of the table is the root sum square of all the uncertainty contributions in the outer right column.The grey part of the table is not filled in.

Table 2.1: Schematic of an ordered arrangement of the quantities, estimates, standard uncertainties, sensitivity coefficients and uncertainty contributions used in the uncertainty analysis of a measurement.

Quantity X_i	Estimate x_i	Standard uncertainty $u(x_i)$	Sensitivity coefficient c_i	Contribution to the standard uncertainty $u_i(y)$
X_1	x_1	$u(x_1)$	c_1	$u_1(y)$
X_2	x_2	$u(x_2)$	c_2	$u_2(y)$
:	:	:	:	.
X_N	x_N	$u(x_N)$	c_N	$u_N(y)$
Y	**y**			**u(y)**

3. EXPANDED UNCERTAINTY OF MEASUREMENT

3.1 Within EAL it has been decided that calibration laboratories accredited by members of the EAL shall state an expanded uncertainty of measurement U, obtained by multiplying the standard uncertainty u(y) of the output estimate by a coverage factor k,

$$U=ku(y) \qquad (3.1)$$

In cases where a normal distribution can be attributed to the measured and the standard uncertainty associated with the output estimate has sufficient reliability, the standard coverage factor k=2 shall be used.The assigned expanded uncertainty corresponds to a coverage probability of approximately 95%.These conditions are fulfilled in the majority of cases encountered in calibration work.

3.2 The assumption of a normal distribution cannot always be easily confirmed experimentally.However ,in the cases where several (i.e.$N\geq 3$) uncertainty components, derived from well-behaved probability distributions of independent quantities, e.g normal distributions or rectangular distributions, contribute to the standard uncertainty associated with the output estimate by comparable amounts, the conditions of the Central Limit Theorem are met and it can be assumed to a high degree of approximation that the distribution of the output quantity is normal.

3.3 The reliability of the standard uncertainty assigned to the output estimate is determined by its effective degrees of freedom the reliability criterion is always met if none of the unsertainty contributions is obtained from a type A evaluation based on less than ten repeated observations.

3.4 If one of these conditions(normality or sufficient reliability) is not fulfilled ,the standard coverage factor k=2 can yield an expanded uncertainty corresponding to the same coverage probability of less than 95%.In these cases,in order to ensure that a value of the of the expanded uncertainty is quoted corresponding to the same coverage probabilities in the normal case, other procedures have to be followed.The use of approximately the same coverage probability is essential whenever two results of measurement of the same quantity have to be compared, e.g. when evaluating the results of an interlaboratory comparison or assessing compliance with a specification.

3.5 Even if a normal distribution can be assumed, it may still occur that the standard uncertainty associated with the output estimate is of insufficient reliability.

3.6 For the remaining cases, i.e. all cases where the assumption of a normal distribution cannot be justified, information on the actual probability distribution of the output estimate must be used to obtain a value of the coverage factor k that corresponds to a coverage probability of approximately 95%.

4. CORRELATED INPUT QUANTITIES

4.1 If two input quantities X_i and X_k are known to be correlated to some extent-i.e if they are dependent on each other in one way or another-the covariance associated with the two estimates x_i and x_k

$$u(x_i,x_k)=u(x_i)u(x_k)r(x_i,x_k) \quad (i \neq k) \tag{4.1}$$

has to be considered as an additional contribution to the uncertainty.The degree of correlation is characterized by the correlation coefficient $r(x_i,x_k)$ (where $(i \neq k)$ and $|r| \leq 1$)

4.2 In the case of n independent pairs of simultaneously repeated observations of two quantities P and Q the covariance associated with the arithmetic means $\overline{p}$ and $\overline{q}$ is given by

$$s(\overline{p},\overline{q})=\frac{1}{n(n-1)}\sum_{j=1}^{n}\left(p_j-\overline{p}\right)\left(q_j-\overline{q}\right) \tag{4.2}$$

and by substitution r can be calculated from equation(4.1)

4.3 For influence quantities any degree of correlation has to be based on experience.When there is correlation, equation (2.1) has to be replaced by

$$u^2(y)=\sum_{i=1}^{N}c_i^2u^2(x_i)+2\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}c_ic_ku(x_i,x_k) \tag{4.3}$$

where c_i and c_k are the sensitivity coefficients defined by equation (2.3) or

$$u^2(y)=\sum_{i=1}^{N}u_i^2(y)+2\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}u_i(y)u_k(y)r(x_i,x_k) \tag{4.4}$$

with the contributions $u_i(y)$ to the standard uncertainty of the output estimate y resulting from the standard uncertainty of the input estimate x_i given by equation (2.2).It should be noted that the second summation of terms in equation (4.3) or (4.4) may become negative in sign.

4.4 In practice, input quantities are often correlated because the same physical reference standard, measuring instrument, reference datum, or even measurement method having a significant uncertainty is used in the evaluation of their values.Without loss of generality, suppose that two input quantities X_i and X_2 estimated by x_1 and x_2 depend on the set of independent variables Q_l (l=1,2,…..L)

$$X_i=g_1(Q_1,Q_2,\dots Q_l) \tag{4.5}$$

$$X_2=g_2(Q_1,Q_2,\dots Q_l)$$

although some of these variables may not necessarily appear in both functions.The estimates x_1 and x_2 of the input quantities will be correlated to some extent, even the estimates $q_l(l=1,2,\ldots,L)$ are ucorrelated.In that case the covariance $u(x_1, x_2)$ associated with the estimates x_1 and x_2 is given by

$$u(x_1, x_2)= \sum_{l=1}^{L} c_{1l} c_{2l} u^2\left(q_l\right) \qquad (4.6)$$

where c_{1l} and c_{2l} are the sensitivity coefficients derived from the functions g_1 and g_2 in analogy to equation (2.3).Because only those terms contribute to the sum for which the sensitivity coefficients do not vanish, the covariance is zero if no variable is common to functions g_1 and g_2.The correlation coefficient $r(x_1, x_2)$ associate with the estimates x_1 and x_2 is determined from equation (4.6) together with equation (4.1).

4.5 The following example demonstrates correlation's which exist between values attributed to two artefact standards that are calibrated against the same reference standard.

Measurement problem

The two standards X_1 and X_2 are compared with the reference standard Q_s by means of a measuring system capable of determining a difference z in their values with an associated standard uncertainty u(z).The value q_s of the reference standard is known with standard uncertainty $u\left(q_s\right)$

Mathematical model

The estimates x_1 and x_2 depend on the value q_s of the reference standard and the observed differences z_1 and z_2 according to the relations

$$\begin{aligned} x_1 &= q_s - z_1 \\ x_2 &= q_s - z_2 \end{aligned} \qquad (4.7)$$

Standard uncertainties and covariance's

The estimate z_1,z_2 and q_s are supposed to be uncorrelated because they have been determined in different measurements.The standard uncertainties are calculated from equation (2.2) and the covariance associated with the estimates x_1 and x_2 is calculated from equation (4.6), assuming that

$$\begin{aligned} u\left(z_1\right) &= u\left(z_2\right) = u(z) \\ u^2\left(x_1\right) &= u^2\left(q_s\right) + u^2(z) \\ u^2\left(x_2\right) &= u^2\left(q_s\right) + u^2(z) \\ u\left(x_1, x_2\right) &= u^2\left(q_s\right) \end{aligned} \qquad (4.8)$$

The correlation coefficient deduced from these results is

$$r\left(x_1, x_2\right) = \frac{u^2\left(q_s\right)}{u^2\left(q_s\right) + u^2(z)} \qquad (4.9)$$

Its value ranges from 0 to +1 depending on the ratio of the standard uncertainties $u(q_s)$ and u(z).

4.6 The case described by equation (4.5) is an occasion where the inclusion of correlation in the evaluation of the standard uncertainty of the measured can be avoided by a proper choice of the model function.Introducing directly the independent variables Q_l by replacing the original variables X_1 and X_2 in the model function f in accordance with the transformation equations (4.5) gives a new model function that does not contain the correlated variables X_1 and X_2 any longer.

4.7 There are cases however, where correlation between two input quantities X_1 and X_2 can not be avoided, e.g. using the same measuring instrument or the same reference standard when determining the input estimates $x_1$and x_2 but where transformation equations to new independent variables are not available.If furthermore the degree of correlation is not exactly known it may be useful to asses the maximum influence this correlation can have by an upper bound estimate of the standard uncertainty of the measured which in the case that other correlation’s have not to be taken into account takes the form:

$$u^2(y) \leq \left(|u_1(y)| + |u_2(y)|\right)^2 + u_r^2(y) \qquad (4.10)$$

with $u_r(y)$ being the contribution to the standard uncertainty of all the remaining input quantities assumed to be uncorrelated.

Note: Equation (4.10) is easily generalized to cases of one or several groups with two or more correlated input quantities.In this case a respective worst case sum has to be introduced into equation (4.10) for each group of correlated quantities.

5. COVERAGE FACTORS DERIVED FROM EFFECTIVE DEGREES OF FREEDOM

5.1 To estimate the value of a coverage factor k corresponding to a specified coverage probability requires that the reliability of the standard uncertainty u(y) of the output estimate y is taken into account.That means taking into account how well u(y) estimates the standard deviation associated with the result of the measurement.For an estimate of the standard deviation of a normal distribution, the degrees of freedom of this estimate, which depends on the size of the sample on which it is based, is a measure of the reliability.Similary, a suitable measure of the reliability of the standard uncertainty associated with an output estimate is its effective degrees of freedom V_{eff}, which is approximated by an appropriate combination of the effective degrees of freedom of its different uncertainty contributions $u_i(y)$.

5.2 The procedure for calculating an appropriate coverage factor k when the conditions of the central limit theorem are comprises the following three steps:

(a) Obtain the standard uncertainty associated with the output estimate according to the step by step.

(b) Estimate the effective degrees of freedom V_{eff} of the standard uncertainty u(y) associated with the output estimate from the Welch-Satterthwaite.

$$V_{eff} = \frac{u^4(y)}{\sum_{i=1}^{N} \frac{u^4(y)}{V_i}} \qquad (5.1)$$

where the $u_i(y)$ (i=1,2….N), defined in equation (2.2), are the contributions to the standard uncertainty associated with the output estimate y resulting from the standard uncertainty associated with the input estimate x_i which are assumed to be mutually statistically independent, and V_i is the effective degrees of freedom of the standard uncertainty contribution $u_i(y)$.

(b) Obtain the coverage factor k from the table of values given as table 5.1 of this annex.This table is based on a t-distribution evaluated for a coverage probability of 95,45%.If V_{eff} is not an integer, which will usually be the case, truncate V_{eff} to the next lower integer.

Table5.1:Coverage factors k for different effective degrees of freedom V_{eff}

V_{eff}	1	2	3	4	5	6	7	8	10	20	50	$\propto$
k	13,97	4,53	3,31	2,87	2,65	2,52	2,43	2,37	2,28	2,13	2,05	2,00

6. STATEMENT OF UNCERTAINTY OF MEASUREMENT IN CALIBRATION CERTIFICATES

6.1 In calibration certificates the complete result of the measurement consisting of the estimate y of the measured and the associated expanded uncertainty U shall be given in the form ($y \pm U$).To this an explanatory note must be added which in the general case should have the following content:

The reported expanded uncertainty of measurement is stated as the standard uncertainty of measurement multiplied by the coverage factor k=2, which for a normal distribution corresponds to a coverage probability of approximately 95%.The standard uncertainty of measurement has been determined in accordance with EAL Publication EAL-R2.

6.2 However, in cases where the procedure of coverage factors derived from effective degrees of freedom has been followed, the additional note should read as follows:

The reported expanded uncertainty of measurement is stated as the standard uncertainty of measurement multiplied by the coverage factor k=XX, which for a t-distribution with V_{eff}=YY effective degrees of freedom corresponds to a coverage probability of approximately 95%.The standard uncertainty of measurement has been determined in accordance with EAL Publication EAL-R2.

6.3 The numerical value of the uncertainty of measurement should be given to at most two significant figures.The numerical value of the measurement result should in the final statement normally be rounded to the least significant figure in the value of the expanded uncertainty assigned to the measurement result.For the process of rounding , the usual rules for rounding of numbers have to be used.However, if the rounding brings the numerical value of the uncertainty of measurement down by more than 5%, the rounded up value should be used.

7. CONCLUSION

Respecting the importance set down principles of and requirements on the evaluation of the uncertainty of measurement in calibration which are appropriate in the most fields of physical measurements appeared need of publishing document which treat this problem.The document establishes general rules for evaluating and expressing uncertainty in measurement.Establishes the document of this type is extremely important.Although, this document concentrates on the method most most suitable for the measurements in calibration laboratories and describes an unambiguous and harmonized way of evaluating and stating the uncertainty of measurement, its easy to apply it in various physical measurements.It's the most contribution is defines the best measurement capability as the smallest uncertainty of measurement that a laboratory can achieve within its scope of accreditation.

8. LITERATURE

/1/ Guide to the Expression of Uncertainty in Measurement ,first edition, 1993, corrected and reprinted 1995, International Organization for Standardization (Geneva, Switzerland)

/2/ International Standard ISO 3534-1 Statistics –Vocabulary and Symbols-Part I: Probability General Statistical Terms, first edition, 1993, International Organization for Standardization (Geneva, Switzerland)

/3/ International Vocabulary of Basic and General Terms in Metrology, second edition, 1993, International Organization for Standardization (Geneva, Switzerland)